Math 171: Writing in the Major
The Mathematical Foundations of the Fourier Series

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1 Introduction

Understanding the behavior of repeated complicated signals (such as one sent through a communication line) and the oscillation of quantum-sized particles have been at the forefront of physics and engineering in the last century. These two areas of interest are closely related and made quantitatively examinable through the *Fourier series*. Originally devised to solve the heat equation by its namesake, the Fourier series introduces an approach to perfectly represent any periodic signal as the sums of sinusoidal basis functions of varying frequencies. The study of the Fourier series has proved fruitful in a wide array of applications, and can be credited for much of the information age and modern day signal processing. Harmonic analysis is the branch of mathematics devoted to understanding the Fourier series (alongside other methods of decomposing functions into different bases) and its implications on continuous and discrete domains.

The goal of this paper is to provide a mathematical intuition for the development of the Fourier series with the assistance of rigorous proofs in the context of real analysis to understand its behavior. After going over basic definitions, the convergence of the Fourier series will be examined using the Dirichlet kernel. Finally, an exploration of the Fourier series in the realm of electrical engineering and applications will be presented.

2 Definitions

The Fourier series can be described by equations for **synthesis** and **analysis**.

Definition 2.1. *Given a 2π-periodic function $f(x)$, we can calculate its Fourier series as follows*

$$S_N(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{1}$$

The synthesis equation, presented above, shows that any 2π-periodic function can be separated into a linear combination of sines and cosines, along with a constant term a_0 (this is also known as a *trigonometic polynomial*). It is useful to think of a_0 as a zero frequency term that equals the vertical offset from the origin of the original function. However, in order to reconstruct a signal, we must first obtain its Fourier coefficients.

Definition 2.2. *The coefficients a_0, a_m, and b_m can be deduced through the use of the analysis equations*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \tag{2}$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)dx, \tag{3}$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(mx)dx \tag{4}$$

for $m \geq 1$, given that the integrals are defined. This is the case if $f(x)$ is piecewise continuous and bounded.

For Equation 2, notice that it is computing the integral of $f(x)$ over a 2π interval and then dividing it by 2π; this simply finds the average value over a period. In other words, it calculates the constant shift of our signal with respect to the variation of the function.

Equations 3 and 4 are best understood from a linear algebraic perspective. Recall that we can define an inner product over $L^2([-\pi, \pi])$ as $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$: this satisfies all of the desired properties of an inner product. Note that $\sin(mx)$ and $\cos(mx)$ are orthogonal basis functions over $L^2([-\pi, \pi])$ [2]. By calculating the inner product of $f(x)$ with these basis functions over our domain, we deduce the projection of our original function onto each basis; these projections are our Fourier coefficients, and make sense in the linear combination presented in 1.

3 Convergence of the Fourier Series

We're interested in closely examining how the Fourier series converges to our original function in the limit. An interesting first step is to examine the limiting behaviour of the Fourier coefficients described above.

Proposition 3.1. The Riemann-Lebesgue Lemma: *if f is 2π-periodic and continuous on $[-\pi, \pi]$, then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |b_n| = 0$.*

Intuitively, we can expect the Fourier series coefficients to become infinitesimally small as we add more and more sinusoids of increasing frequency. In other words, we can get a reasonably good approximation of a given 2π-function without having an infinite number of terms. This is important for practical purposes where maintaining an infinite number of terms (say, in a computer) is infeasible. We can get away with much less, at a small cost of the function's fidelity.

We begin our exploration by constructing a sequence of functions necessary to prove the pointwise convergence of the Fourier series.

Definition 3.2. *We now define a sequence of functions known as the **Dirichlet kernel**,*

$$D_N(x) = \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N} \cos(nx)\right). \tag{5}$$

The Dirichlet kernel can be understood as a way to estimate the Dirac delta function (at large values of N) in a continuous form, and thus suited for mathematical analysis. In specific, the Dirichlet kernel enables us to construct an integral representation of the Fourier series (shown in Proposition 3.3 below) which is necessary to prove pointwise convergence.

Proposition 3.3. *By definition of the Dirichlet kernel, we can show*

$$\int_{-\pi}^{\pi} D_N(x)dx = 1. \tag{6}$$

Proof. Expanding the summation and directly evaluating the integral gives us

$$\begin{aligned}\int_{-\pi}^{\pi} D_N(x)dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}x + \cos(x) + \cos(2x) + \cdots + \cos(Nx)\right) \\ &= \frac{1}{\pi}\left(\frac{1}{2}x + \sin(x) + \frac{1}{2}\sin(2x) + \cdots + \frac{1}{N}\sin(Nx)\right)\Bigg|_{-\pi}^{\pi}.\end{aligned} \tag{7}$$

Note that all of the $\sin(nx)$ terms go to 0 since $\sin(n\pi) = \sin(n(-\pi)) = 0$. Thus, all we're left with is

$$\frac{1}{2\pi}x\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi - (-\pi)) = 1 \tag{8}$$

which concludes the proof. □

We now seek to derive an integral representation of the Fourier series. Interestingly, this can easily be done by calculating the convolution of the original periodic function and the Dirichlet kernel. For those with a signal processing background, recall that convolving any function with the Dirac delta function always yields the original function. The intuition here is identical.

Proposition 3.4. *The convolution of the Dirichlet kernel, $D_N(x)$, with a 2π-periodic function $f(x)$ results in the n^{th} degree Fourier series approximation to f,*

$$S_N(x) = (D_N * f)(x) = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy, \tag{9}$$

where $S_N(x)$ was initially presented in Equation (1).

Proof. Substitute the Fourier coefficients a_0, a_m, and b_m into $S_N(x)$, the partial sums of the Fourier series, which results in

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n=0}^{N} (\cos(nx)\cos(ny) + \sin(nx)\sin(ny)) \right] f(y)dy. \tag{10}$$

Note that we are using formulae for the Fourier coefficients with respect to $f(y)$ to create a distinction between the coefficients and the basis functions. Recall the trigonometric identity $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, which we use to simplify the summation to

$$S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[1 + 2 \sum_{n=0}^{N} (\cos(n(x-y))) \right] f(y)dy. \tag{11}$$

Taking the $\frac{1}{2\pi}$ term outside the integral into account, we can recognize the Dirichlet kernel $D_N(x - y)$, such that

$$S_N(x) = \int_{-\pi}^{\pi} D_N(x-y) f(y)dy, \tag{12}$$

thus completing our proof. □

Another form of the Dirichlet kernel is proved below. In particular, this definition is useful because it gives us an explicit formula for the Dirichlet kernel without a summation. This form is the periodic sinc function, commonly used in digital signal processing for filtering.

Proposition 3.5. *An equivalent form of the Dirichlet kernel is*

$$D_N(x) = \frac{1}{2\pi} \frac{\sin((N + \frac{1}{2})x)}{\sin(x/2)}. \tag{13}$$

However, note that this fraction is undefined when the denominator is 0*, and that might reduce the utility of this form.*

Proof. Start with the Dirichlet kernel defined in Equation (5). Multiply both sides by $\sin(x/2)$ and apply the trigonometric identity $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$:

$$\sin(\frac{x}{2}) D_N(x) = \frac{1}{2\pi} \left[\sin(\frac{x}{2}) + \sum_{n=1}^{N} [\sin((n + \frac{1}{2})x) - \sin((n - \frac{1}{2})x)] \right]. \tag{14}$$

The righthand side above is a telescoping series where all terms except for the last cancel (including the $\sin(x/2)$), leaving us with

$$\sin(\frac{x}{2}) + \sum_{n=1}^{N} [\sin((n + \frac{1}{2})x) - \sin((n - \frac{1}{2})x)] = \sin((N + \frac{1}{2})x). \tag{15}$$

We conclude this proof by dividing across this expression by $\sin(x/2)$ to get back to $D_N(x)$. Thus,

$$D_N(x) = \frac{1}{2\pi}\frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}. \tag{16}$$

□

We are ready to prove the pointwise convergence of the Fourier series.

Proposition 3.6. *If f is continuously differentiable on $[-\pi, \pi]$, then $\{S_N(x)\}_{N=1}^{\infty}$ converges to $f(x)$ pointwise on $[-\pi, \pi]$.*

Proof. We start by applying a change of variables on Equation (9) where $z = y - x$. Then, we have

$$S_N(x) = \int_{-\pi}^{\pi} f(y)D_N(x-y)dy = \int_{-\pi-x}^{\pi-x} f(z+x)D_N(-z)dz. \tag{17}$$

By exploiting evenness of the Dirichlet kernel, we get $D_N(-z) = D_N(z)$. By Equation (5), we know the Dirichlet kernel is composed of constants (which don't affect evenness) and a sum of cosines, which are known to be even. Thus, the kernel itself must also be even. Moreover, we can change the limits of the integral back to $-\pi$ and π since everything under the integral is 2π-periodic. Finally, we rename z to y to get a form that uses the same variable names as Equation (9) to get

$$S_N(x) = \int_{-\pi}^{\pi} f(x+y)D_N(y)dy. \tag{18}$$

Then, we pick a point $x \in [-\pi, \pi]$ at where we will prove convergence. We then subtract the original function from our integral representation of the Fourier series shown above and use the explicit form of the Dirichlet kernel:

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} [f(x+y) - f(x)]D_N(y)dy = \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\sin[(N+\frac{1}{2})y]dy, \tag{19}$$

where the first step is justified by the integral in Equation (6), and

$$g_x(y) = \frac{f(x+y) - f(x)}{2\sin(y/2)}, y \neq 0, g_x(0) = f'(x). \tag{20}$$

We also let $g_x(y) = f'(x)$ wherever the denominator is zero (at 2π intervals from 0). Thus, $g_x(y)$ must be continuous due to the differentiability of f on $[-\pi, \pi]$.

We can apply the trigonometric identity $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$ to $\sin[(N+\frac{1}{2})y]$ to get

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\sin[(N+\frac{1}{2})y]dy &= \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)(\sin(Ny)\cos(y/2)+\cos(Ny)\sin(y/2))dy \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}[g_x(y)\cos(y/2)]\sin(Ny)dy \\ &\quad + \frac{1}{\pi}\int_{-\pi}^{\pi}[g_x(y)\sin(y/2)]\cos(Ny)dy.\end{aligned} \tag{21}$$

We can recognize this as nothing but the sum of Fourier coefficients, a_N with respect to $c_x(y) = g_x(y)\sin(y/2)$ and b_N with respect to $d_x(y) = g_x(y)\cos(y/2)$, as $c_x(y)$ and $d_x(y)$ are both 2π-periodic.

Taking the limit $N \to \infty$, we note that b_N and a_N ultimately converge to zero by the Riemann-Lebesgue lemma, which implies $S_N(x) - f(x) = 0$ or $S_N(x) = f(x)$. Thus, $S_N(x)$ converges pointwise to $f(x)$. □

We cannot show the Fourier series converges uniformly. Intuitively, both uniform and pointwise convergence states that the partial sums at each point in the interval will eventually converge to the value of $f(x)$ at that point. However, they differ in that for pointwise convergence, partial sums may not converge at the same rate at each x in the domain. This provides insight to the Gibbs phenomenon, which will be discussed in the final section of this paper.

4 An Exploration: The Gibbs Phenomenon and Filter Design

The Fourier Series is central to the fields of communication, information theory, and signal processing. In practice, engineers build *lowpass filters* to bandlimit signals from noise and other artifacts. To bandlimit a signal is to simply truncate its Fourier series representation (since higher frequencies are generally associated with noise, also known as *aliasing*). An analog lowpass filter can be constructed with simple electronic devices, such as resistors and capacitors, whose resistance and capacitance determine the bandlimiting frequency and gain. These two characteristics of a filter can be formulated into a *transfer function.*

In sum, a digital filter separates an input signal into individual frequencies, multiplies each of them by a gain factor given by the transfer function, and finally adds up the resulting terms to give a output signal.

Issues with filters arise when considering the implication of bandlimiting signals with jumps or discontinuities. Take for example, the 2π-periodic step function $H(x)$, which equals $\pi/2$ when $0 < x < \pi$ and $-\pi/2$ for $-\pi < x < 0$. Intuitively, we cannot represent this type of function with a finite summation of sinusoidal functions (for example, after lowpass filtering the step function above at an arbitrary bandlimiting frequency). Except in the limit, it is impossible to construct a discontinuity from continuous functions. Hence, we will expect oscillatory behavior and overshoot in the neighborhood of the discontinuity. This is the Gibbs phenomenon. See Figure 1 for a visual representation.

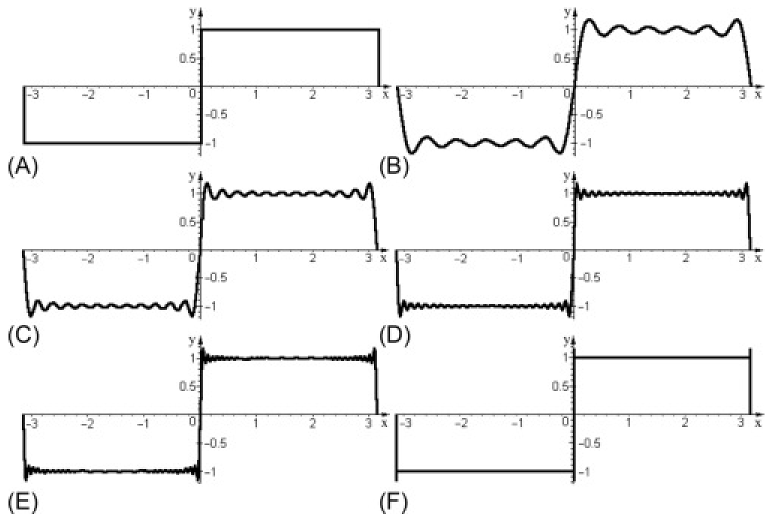

Figure 1: Approximation of a square wave function by a Fourier series: (a) square wave function (described above); partial sums: (b) $k = 6$, (c) $k = 12$, (d) $k = 24$, (e) $k = 36$, (f) $k \to \infty$. Taken from [3]

Intuition for the Gibbs phenomenon can come from the result that presented in Equation **??**. If we take N to be a finite value, the Dirichlet kernel takes on a a periodic sinusodal shape with a larger lobe at the origin. Clearly, convolving this oscillating function with a perfect step would lead to a step with some wiggle on the flat parts and overshoot at the discontinuity. In the limit, recall the Dirichlet kernel converges to a periodic Dirac delta function, and convolution with the Dirac delta would return the original step function. For those unfamiliar with the Dirac delta, it is the identity for convolution.

There is a nice theorem in analysis that helps with furthering this understanding.

Proposition 4.1. *Let (M, d_M) and (N, d_N) be metric spaces. Let $\{f_n\}$ be a sequence of mappings from M to N such that:*

1. *For all $n \in \mathbb{N}$: f_n is continuous at every point of M*
2. *$\{f_n\}$ converges uniformly to f*

Then, f (the limit function) is continuous at every point of M.

The proof for this theorem is fairly straightforward: it uses the application of the triangle inequality and the epsilon-delta definition of uniform convergence to prove continuity of f on M [4].

Let $\{f_n\}$ be defined as a sequence of our partial sums, $S_N(x)$, for the Fourier series. We know that as $N \to \infty$, $S_N(x) \to f(x)$.

We cannot guarantee the limit function, f, to be continuous at every point when calculating a Fourier series expansion. Take the step signal above as an example. Logically, this implies that either (1) or (2) in the proposition are false. However, we know (1) to be true; after all, $\sin(x)$ and $\cos(x)$ are in $\mathcal{C}^\infty$. This means (2) must be false for the Fourier series. Thus, we cannot guarantee uniform convergence. We can at most prove pointwise convergence as we did in the previous section. For this reason, we should expect to see overshoot/undershoot in the region of the discontinuity. Even if we add more terms to our Fourier expansion, this theorem suggests that we should still expect a nonzero overshoot.

An interesting observation is that as the number of terms in the Fourier expansion of a discontinuous function increases, the Gibbs ringing reduces in width. However, it converges to a fixed height. In fact, we can quantify the exact value of Fourier series overshoot at a jump in a square wave to be approximately 9% on both ends.

Definition 4.2. *The* ***Wilbraham-Gibbs*** *constant is*

$$G = \int_0^\pi \frac{\sin(t)}{t} dt = 1.8519370\ldots$$

which can be used to calculate overshoot at discontinuities of various Fourier series using the formula

$$\textit{overshoot} = \frac{G - \textit{limit value at } 0^+}{\textit{total jump}}.$$

For our square wave example above,

$$\textit{overshoot} = \frac{G - \pi/2}{\pi} = 0.0894 \approx 9\%.$$

The proof for this definition involves creating a N^{th} partial Fourier expansion for the function, finding the location of the overshoot, and then cleverly identifying a Riemann sum that corresponds to the integral shown above in the definition. See [5] for more details on the actual proof.

This discussion examined the Gibbs phenomenon using filter design as a motivating example. To round off our discussion on filter design, it is important to recognize that the Gibbs phenomenon cannot be ignored in many cases. The question arises: how can we bandlimit functions while controlling the Gibbs phenomenon? The reason for the Gibbs artifacts lies in sudden truncation for the Fourier series by a filter. We can use a variety of approaches to go about this.

One approach is to use the Fejér kernel instead of the Dirichlet kernel.

Definition 4.3. *The* ***Fejér kernel*** *is*

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x),$$

where $D_k(x)$ is the k^{th} order Dirichlet Kernel.

Unlike the Dirichlet kernel, the Fejér kernel is strictly positive. Moreover, using this kernel results in uniform convergence, and thus, no offshoot. This may be a more appealing approach for certain applications, at the cost of significantly more computation.

References

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